First Order Logic

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1 Introduction

First order logic (FOL) is a reasoning framework that encapsulates most real-world use-cases for reasoning about problems. A formula in the logic is an expression that presents a (true-false) statement about the system or problem. A very simple example that is true about the world:

The sun rises in the east and sets in the west.

We all know that this is a valid statement no matter where we are on earth (except the poles where the sun does not change position). The above statement can be posited in the framework of First order logic. We first define predicates *rises(x, y)* and *sets(x, y)*, where x is an object and y is a direction from the set {`east, west, north, south`}. The statement is posited as:

rises(sun, east) and sets(sun, west).

The statement succintly represents the valid statement presented above in the framework of First order logic. The benefit of representing ideas as logic statements is that it allows further inferences and judgements, which may not be possible otherwise. The simplest form of logic is that of Propositional Logic. The logic assigns a `true` and `false` value to each variable within a Boolean formula, which is used to obtain the truth value for the formula. Unlike Propositional logic, First order logic allows predicate symbols which are functions from arbitrary domains to Boolean values (for example *rises* and *sets* above). The predicates can be used to extend the domain of objects represented by the formula.

The rich expressivity of the First order logic comes with additional costs. Proving satisfiability of a First order logic formula (finding a feasible assignment that satisfies a given formula in FOL) is undecidable (not feasible in general). Yet, a large number of First order logic formulae exist around us in the form of constraints and problems that we would like to tackle in everyday life. A saving grace in this complexity relies on various Decision Theories which are easy to address in practice. Each Decision theory comes with a set of basic constraints or *axioms*, thereby, making reasoning about the overall problem simpler. We discuss a few theories below.

2 Theory of equality

We consider a very simple theory of equality, which consists of predicates that are equality constraints between values and their negations. Formulae in the theory of equality consist of statements about the equality of various objects / entities in the world. Consider the following statement in the theory of equality:

$$\text{Age(Ram)} = \text{Age(Sita) and Age(Sita)} = \text{Age(Geeta)} \Rightarrow \text{Age(Ram)} = \text{Age(Geeta)}.$$

The above statement states that if ages of Ram and Sita are equal and the ages of Sita and Geeta are equal then the age of Ram is equal to that of Geeta. This is a simple theory and does not require complex reasoning to validate. We simply collect all objects that are equal in an *equivalence class*. Any two objects / entities are equal from an equivalence class. Further if we have information that two entities from classes A and B are not equal, then all objects from A are distinct from that in B. Collections of equal entities can be formed easily and allow us to solve or prove statements in the theory.

3 Theory of Rationals

Next, we consider the theory of rationals, where we consider arithematic constraints along with equality constraints. The arithematic constraints enable reasoning about quantities, often appears in various situations like marketplace where goods are traded in exchange for a monetary value, in offices where accounting of finances across the company need to be done to ensure parity, between friends when settling shared expenses, etc. Such reasoning can often present non-trivial constraints, requiring a richer theory of rationals to decide a resolution. Here's an example of a constraint in the market, where the buyer is willing to pay a certain amount, while the seller is expecting a minimum value for each of their goods.

$$\text{SellPrice(TShirt)} > 10 \text{ and SellPrice(Shorts)} > 20.$$
$$\text{SellPrice(2TShirt + 1 Short)} == \text{SellPrice(3TShirt)}.$$
$$\text{BuyPrice(2TShirt + 2 Short)} < 55.$$

In the above example, there is a resolution possible, where two TShirts and a Shorts pair is sold for 33, and another pair of Shorts is sold for 21. This would give the desired final settlement between the seller and the buyer.

4 Interaction between theories

A general statement in First order logic is often not limited to a single theory, instead consists a mix of multiple theories. This poses a challenge when we are interested in solving problems and tackling constraints from different theories. First order logic provides a neat solution to the above issue. The logic allows one to separate constraints from different theories into distinct sets. Further, inference for each set can be performed separately without any interference from the other theories. This is effectively addressed in the logic by requiring one to only send equality or disequality of entities within a set to the other constraint sets. It has been shown that equality constraints are sufficient to prove the correctness of multiple constraints. To understand this further, consider the following example:

$$Age(Ram) == Age(Sita) \text{ and } Geeta == Sita \text{ and}$$

$$Age(Shyam) == Age(Geeta) + 1 \text{ implies}$$

$$Age(Shyam) == Age(Ram) + 1.$$

First from the theory of equality we infer that *Sita* and *Geeta* are the same person, and therefore, age of Sita is same as age of Geeta. We know that age of Ram is equal to age of Sita. Therefore, age of Ram is equal to age of Geeta. We can pass this equality to the constraint Age(Shyam) == Age(Geeta) + 1. This gives us the desired final result.

In general, we can solve constraints where multiple theories are present by making separate inferences in each theory and passing the result of the inferences to other theories which provide us the desired result.

5 Conclusion

First order logic is a powerful framework for reasoning about systems of constraints. It allows one to express generic problems in real-world and obtain their solution by making inferences in specific theories and then passing the inferences to constraints in other theories. This general power of the framework allows one to solve complex constraints and obtain solutions where other frameworks might fail (due to lack of expressivity or specificity). It truely represents a universal system that can encompass a large number of theories and yet be feasible to solution for many problems.

6 References

1. Calculus of Computation. Aaron Bradley and Zohar Manna. Springer 2007.